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Duality and Character Values of Finite Groups of Lie Type

DEAN ALVIS

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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INTRODUCTION

The purpose of this paper is to show how the values of the dual of a character of a finite group of Lie type can be computed on semisimple classes, given sufficient information about the original character.

In the first approach, the values of a character χ on elements su with (fixed) semisimple part s are related to the value of the dual χ^* at s. To be specific, $\sum \chi(su) = \mathrm{St}(s)\,\chi^*(s)$, where St is the Steinberg character. The special case in which χ is a component of the permutation character 1_B^G and $s = 1$ was established by Kawanaka in [15].

A second method can be used for the group of rational points of a connected, reductive algebraic group defined over a finite field with q elements. It is shown that $\chi^*(s) = \mathrm{St}(s)\,\chi^{\#}(s)$, where, roughly speaking, $\chi^{\#}(s)$ is $\chi(s)$ with q replaced by $1/q$ (see Section 3). When $s = 1$ and χ is a component of 1_B^G, this formula coincides with a well-known formula involving generic degrees (see Green [12] and Curtis [9]).

In Section 4, both approaches are applied to prove, for large q, a conjecture of I. Macdonald (see [21, 6.11]). Special cases of this conjecture have been proved in Kawanaka [15] and Springer [22].

Most of the results in this paper were announced in [2] and proved in the author's dissertation [3].

1. PRELIMINARY RESULTS

If H is a finite group, irr H will denote the set of irreducible characters of H, char H the virtual characters of H, and cf H the class functions on H, all over the field $\mathbb{C}$ of complex numbers. Let $(\,,\,)_H$, or simply $(\,,\,)$, denote the usual Hermitian form on cf H; $(f, g) = |H|^{-1} \sum_{h \in H} f(h)\,\overline{g(h)}$.

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Let G be a finite group with split (B, N)-pair of characteristic p (see [6, 8, 20]), $G = \langle B, N \rangle$. Let (W, R) be the Coxeter system for G; R is a set of distinguished generators for the Coxeter group $W = N/T$, $T = B \cap B$. For $J \subseteq R$ let W_J be the subgroup of W generated by J. Then $P_J = BW_JB$ is a standard parabolic subgroup of G, with Levi decomposition $P_J = L_J V_J$, $V_J = O_p(P_J)$. The Levi factor L_J has a split (B, N)-pair of characteristic p, with Coxeter system (W_J, J). If $K \subseteq J$, then $P_{J,K} = L_J \cap P_K$ is a standard parabolic subgroup of L_J, with Levi decomposition $P_{J,K} = L_K V_{J,K}$, $V_{J,K} = L_J \cap V_K$. Also, V_K is the semidirect product $V_K = V_J V_{J,K}$.

For $K \subseteq J \subseteq R$ and $\varphi \in \operatorname{irr} L_K$, let $\tilde{\varphi}$ be the extension of φ to $P_{J,K}$ via the projection $P_{J,K} \to L_K \cong P_{J,K}/V_{J,K}$. Thus $\tilde{\varphi}(lv) = \varphi(l)$ for $l \in L_K$, $v \in V_{J,K}$. We define $I_K^J \varphi = \operatorname{Ind}_{P_{J,K}}^{L_J} \tilde{\varphi}$. Note that I_K^J extends to linear maps I_K^J: $\operatorname{char} L_K \to \operatorname{char} L_J$ and I_K^J: $\operatorname{cf} L_K \to \operatorname{cf} L_J$. We will write I_K instead of I_K^R when $J = R$.

Assume now that $K \subseteq J \subseteq R$ and $\chi \in \operatorname{irr} L_J$. Then define $T_K^J \chi = \sum_{\varphi \in \operatorname{irr} L_K} (\chi, I_K^J \varphi)_{L_J} \varphi$; $T_K^J \chi$ is called the truncation of χ to L_K (Curtis [9]). We will write T_K instead of T_K^R when $J = R$. If χ is the character of the $\mathbb{C}L_J$-module M, then $T_K^J \chi$ is the character of L_K afforded by the subspace of $V_{J,K}$-invariants of M. Note that T_K^J extends to linear maps T_K^J: $\operatorname{char} L_J \to \operatorname{char} L_K$ and T_K^J: $\operatorname{cf} L_J \to \operatorname{cf} L_K$. If $f \in \operatorname{cf} L_J$, then the value of $T_K^J f$ at $l \in L_K$ is given by

$$T_K^J f(l) = |V_{J,K}|^{-1} \sum_{v \in V_{J,K}} f(lv). \tag{1.1}$$

The next two lemmas follow easily from the remarks above and will be stated without proof.

(1.2) Lemma. *If $K \subseteq J \subseteq R$, $f \in \operatorname{cf} L_J$ and $g \in \operatorname{cf} L_K$, then $(f, I_K^J g)_{L_J} = (T_K^J f, g)_{L_K}$.*

(1.3) Lemma. *If $K \subseteq J \subseteq R$, then $I_K = I_J I_K^J$ and $T_K = T_K^J T_K$.*

Let f be a class function on G. Following Curtis [9], we define the dual f^* of f to be the class function $f^* = \sum_{J \subseteq R} (-1)^{|J|} I_J T_J f$.

(1.4) Theorem (Curtis [9, Theorem 1.3]). *If $J \subseteq R$ and $f \in \operatorname{cf} G$, then $(T_J f)^* = T_J(f^*)$.*

The proof of the following theorem, which is outlined in [1], will be included for the sake of completeness. This theorem has also been proved independently by N. Kawanaka (private communication).

(1.5) Theorem. *The duality map $f \to f^*$ is an isometry on $\operatorname{cf} G$ of order two. Thus $\chi^{**} = \chi$ and $\pm\chi^* \in \operatorname{irr} G$ if $\chi \in \operatorname{irr} G$.*

Proof. Let $f, g \in \operatorname{cf} G$. Using the definition for f^*, we see that (f^*, g) is equal to

$$\sum_{J \subseteq R} (-1)^{|J|} \sum_{\varphi \in \operatorname{irr} L_J} (f, I_J\varphi)(I_J\varphi, g). \tag{1.6}$$

The symmetry of (1.6) shows that $(f^*, g) = (f, g^*)$. It remains to prove $f^{**} = f$.

We have $f^{**} = \sum_{J \subseteq R} (-1)^{|J|} I_J T_J(f^*) = \sum_{J \subseteq R} (-1)^{|J|} I_J((T_J f)^*)$ by Curtis' result, Theorem 1.4. Expanding $(T_J f)^*$, we obtain

$$f^{**} = \sum_{J \subseteq R} (-1)^{|J|} \sum_{K \subseteq J} (-1)^{|K|} I_J I_K^J T_K^J T_J f. \tag{1.7}$$

Finally, applying Lemma 1.3 to (1.7) gives

$$f^{**} = \sum_{K \subseteq R} (-1)^{|K|} \left(\sum_{J \supseteq K} (-1)^{|J|} \right) I_K T_K f. \tag{1.8}$$

For a given $K \subseteq R$, the contribution to the sum in (1.8) is zero unless $K = R$. Therefore $f^{**} = I_R T_R f = f$, which finishes the proof of the theorem.

Let $J \subseteq R$ and assume $\varphi \in \operatorname{irr} L_J$ is cuspidal. That is, $T_K^J \varphi = 0$ when $K \subset J$, $K \neq J$. An immediate consequence of Theorems 1.4 and 1.5 and Lemma 1.2 is that $(I_J\varphi)^* = I_J(\varphi^*) = (-1)^{|J|} I_J\varphi$. If $\chi \in \operatorname{irr} G$ is a component of $I_J\varphi$ of multiplicity m, so that $m = (\chi, I_J\varphi) > 0$, we deduce from Theorem 1.5 that $(-1)^{|J|}\chi^* \in \operatorname{irr} G$, and that $(-1)^{|J|}\chi^*$ is also a component of $I_J\varphi$ of multiplicity m.

2. Duality and Jordan Decomposition

Suppose H is a finite group, $x \in H$, and that p is a prime. There is a unique p-element u of H and a unique p'-element s of H such that $x = su = us$; u (s) is called the p-part (p'-part, respectively) of x. The proof of the following lemma is elementary and will be omitted (see Lusztig [17, Sect. 2]).

(2.1) Lemma. *Let P be a finite group which is a semidirect product $P = LV$, where V is a normal p-subgroup of P. Assume $l \in L$ and $v \in V$. Then the p'-parts of l and lv are conjugate by an element of V.*

Now let $G = \langle B, N \rangle$ be a finite group with split (B, N)-pair of characteristic p. If $x \in G$, the p-part (p'-part) of x will be called the unipotent (semisimple) part of x, and will be denoted by x_{un} (x_{ss}, respectively). The element x will be called unipotent (semisimple) if $x = x_{un}$ ($x = x_{ss}$, respectively).

It will be convenient to use a basis for cf G other than irr G. For $x, y \in G$, define $\rho_x(y) = |Z(x)|$ if x and y are conjugate, and set $\rho_x(y) = 0$ otherwise. Then $\{\rho_x \mid x \in G\}$ is a basis for cf G, and $(f, \rho_x) = f(x)$ for $f \in$ cf G, $x \in G$.

(2.2) LEMMA. *If $x, y \in G$ and $\rho_x^*(y) \neq 0$, then x_{ss} and y_{ss} are conjugate.*

Proof. If $\rho_x^*(y) \neq 0$, then $I_J T_J \rho_x(y) \neq 0$ for some $J \subseteq R$. Thus there are $l \in L_J$ and $v \in V_J$ such that y is conjugate to lv and $(T_J \rho_x)^\sim(lv) = T_J \rho_x(l) \neq 0$. From (1.1) we see that $\rho_x(lu) \neq 0$ for some $u \in V_J$, so that x is conjugate to lu. The proof is completed by applying Lemma 2.1 to $P_J = L_J V_J$.

Let $s \in G$ be semisimple. Denote by $V(s)$ the set of all unipotent elements in $Z(s)$, the centralizer of s.

(2.3) THEOREM. *Let $s \in G$ be semisimple, and assume $f, g \in$ cf G. Then*

$$\sum_{u \in V(s)} f(su)\, g(su) = \sum_{u \in V(s)} f^*(su)\, g^*(su).$$

Proof. It is enough to argue in the case $f = \rho_x$, $g = \rho_y^*$, $x, y \in G$, by Theorem 1.5. It must be shown that

$$\sum_{u \in V(s)} \rho_x(su)\, \rho_y^*(su) = \sum_{u \in V(s)} \rho_x^*(su)\, \rho_y(su),$$

or, equivalently, that

$$\rho_y^*(x) \sum_{u \in V(s)} \rho_x(su) = \rho_x^*(y) \sum_{u \in V(s)} \rho_y(su). \tag{2.4}$$

By Lemma 2.2, both sides of (2.4) vanish unless x_{ss} and y_{ss} are conjugate. We may reduce to the case $x = su_0$, $y = sv_0$, $u_0, v_0 \in V(s)$. An easy calculation shows both sums in (2.4) are equal to $|Z(s)|$. Finally, $\rho_y^*(x) = (\rho_y^*, \rho_x) = (\rho_y, \rho_x^*) = (\rho_x^*, \rho_y) = \rho_x^*(y)$ by Theorem 1.5. This completes the proof of the theorem.

Let St be the Steinberg character of G. Recall that St is an irreducible character of G of degree $|G|_p$, and that $\mathrm{St} = \sum_{J \subseteq R} (-1)^{|J|} \operatorname{Ind}_{P_J}^G 1_{P_J}$ (Curtis [7, Theorem 2]). Thus $\mathrm{St} = 1_G^*$, and St vanishes off the semisimple classes of G. The following three corollaries are therefore special cases of Theorem 2.3.

(2.5) COROLLARY. *Let $s \in G$ be semisimple, and let χ be a character of G. Then $\sum_{u \in V(s)} \chi(su) = \mathrm{St}(s)\, \chi^*(s)$.*

(2.6) COROLLARY. *If χ is a character of G, then* $\sum_{u \in V(1)} \chi(u) = |G|_p \chi^*(1)$.

(2.7) COROLLARY (Steinberg [23]). *If $s \in G$ is semisimple, then* $|V(s)| = \mathrm{St}(s)^2$.

The next result can be thought of as an orthogonality relation and gives another connection between the duality operation $\chi \to \chi^*$ and Jordan decomposition.

(2.8) THEOREM. *Let $s \in G$ be semisimple and $x \in G$ arbitrary. The quantity $\sum_{\chi \in \mathrm{irr}\, G} \chi(x) \overline{\chi^*(s)}$ is equal to $|Z(s)|/\mathrm{St}(s)$ if x_{ss} is conjugate to s and vanishes otherwise.*

Proof. Applying Corollary 2.5 we obtain

$$\sum_{\chi \in \mathrm{irr}\, G} \chi(x) \overline{\chi^*(s)} = \mathrm{St}(s)^{-1} \sum_{u \in V(s)} \sum_{\chi \in \mathrm{irr}\, G} \chi(x) \overline{\chi(su)}.$$

An orthogonality formula for characters ([10, 31.19]) shows the quantity in question vanishes unless x is conjugate to some su, $u \in V(s)$, which occurs only if x_{ss} is conjugate to s. We may assume, therefore, that $x_{ss} = s$. From the above, $\sum_{\chi \in \mathrm{irr}\, G} \chi(x) \overline{\chi^*(s)} = N\,|Z(x)|/\mathrm{St}(s)$, where N is the number of su, $u \in V(s)$, conjugate to x. Since $N = |Z(s) : Z(x)|$, the theorem is proved.

(2.9) COROLLARY. *If $u \in G$ is unipotent, then* $\sum_{\chi \in \mathrm{irr}\, G} \chi(u) \chi^*(1) = |G|_{p'}$. *In particular,* $\sum_{\chi \in \mathrm{irr}\, G} \chi(1) \chi^*(1) = |G|_{p'}$.

3. A SECOND METHOD

In this section we assume $G = \mathbf{G}^F$ is the group of rational points of a connected, reductive algebraic group $\mathbf{G}$ defined over a finite field $\mathbb{F}_q$ with Frobenius endomorphism F. The group $G = \mathbf{G}^F$ has a split (B, N)-pair of characteristic p, where $q = p^e$, with $B = \mathbf{B}_0^F$ for an F-stable Borel subgroup $\mathbf{B}_0$ of $\mathbf{G}$, $T = \mathbf{T}_0^F$ for an F-stable maximal torus $\mathbf{T}_0 \leqslant \mathbf{B}_0$, $N = N(\mathbf{T}_0)^F$ and $W = \mathbf{W}^F$ for $\mathbf{W} = N(\mathbf{T}_0)/\mathbf{T}_0$. Let $\mathrm{rank}_q(\mathbf{G})$ be the dimension of a maximal $\mathbb{F}_q$-split torus in $\mathbf{G}$, and define $\varepsilon_{\mathbf{G}} = (-1)^r$, $r = \mathrm{rank}_q(\mathbf{G})$.

If $\mathbf{T}$ is an F-stable torus in $\mathbf{G}$, then $q^{-1}F$ acts on the character group $X(\mathbf{T})$ as an automorphism of finite order. Let $\lambda_{\mathbf{T}}(u)$ be the characteristic polynomial of $q^{-1}F$ on $X(\mathbf{T})$ ($\lambda_{\mathbf{T}}(u)$ has integer coefficients). The multiplicity of $u - 1$ as a factor of $\lambda_{\mathbf{T}}(u)$ is equal to $\mathrm{rank}_q(\mathbf{T})$. Thus $u^d \lambda_{\mathbf{T}}(1/u) = \varepsilon_{\mathbf{T}} \lambda_{\mathbf{T}}(u)$, where $d = \dim \mathbf{T}$. Also, $\lambda_{\mathbf{T}}(q) = |\mathbf{T}^F|$. If $\mathbf{T}_0 \leqslant \mathbf{B}_0 \leqslant \mathbf{G}$ are as above, then

$|\mathbf{G}^F| = q^{|\Phi^+|} \lambda_{\mathbf{T}_0}(q) P_G(q)$, where Φ is the root system of $\mathbf{G}$ and $P_{\mathbf{G}}(u) = \sum_{w \in \mathbf{W}^F} u^{l(w)}$ is the Poincaré polynomial of $(\mathbf{G}, F)$.

Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$, and let θ: $\mathbf{T}^F \to C^*$ be a linear character. Deligne and Lusztig [11] have constructed, for each such pair $(\mathbf{T}, \theta)$, a virutal character $R_{\mathbf{T}}^{\mathbf{G}}\, \theta$ of $\mathbf{G}^F$. Let $s \in \mathbf{G}^F$ be semisimple, and assume $\chi \in \operatorname{char} \mathbf{G}^F$. Theorem 6.8 and Propositions 7.3 and 7.5 of [11] imply that

$$\chi(s) = \sum_{(\mathbf{T},\theta)} \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{G}} \frac{(\chi, R_{\mathbf{T}}^{\mathbf{G}}\theta)}{(R_{\mathbf{T}}^{\mathbf{G}}\theta, R_{\mathbf{T}}^{\mathbf{G}}\theta)} \frac{\theta^{\mathbf{G}^F}(s)}{\operatorname{St}(s)}, \tag{3.1}$$

where the sum is over the $\mathbf{G}^F$-conjugacy classes of pairs $(\mathbf{T}, \theta)$.

If $\mathbf{T}$ contains a $\mathbf{G}^F$-conjugate of s, define $f_{\mathbf{T},s}(q) = |\mathbf{Z}(s)^{0F}|_{p'}/\mathbf{T}^F|$, interpreted as a polynomial (with integer coefficients) evaluated at q. Also, set

$$g_{\mathbf{T},\theta,\chi,s} = \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{Z}(s)^0} |\mathbf{Z}(s)^F : \mathbf{Z}(s)^{0F}| \frac{(\chi, R_{\mathbf{T}}^{\mathbf{G}}\theta)}{(R_{\mathbf{T}}^{\mathbf{G}}\theta, R_{\mathbf{T}}^{\mathbf{G}}\theta)} \sum \theta t,$$

where $\sum \theta t$ is summed over all $\mathbf{G}^F$-conjugates t of s in $\mathbf{T}^F$. Since $\operatorname{St}(s) = \varepsilon_{\mathbf{G}} \varepsilon_{\mathbf{Z}(s)^0} |\mathbf{Z}(s)^{0F}|_p$, (3.1) implies the following.

$$\chi(s) = \sum_{(\mathbf{T},\theta)} g_{\mathbf{T},\theta,\chi,s} f_{\mathbf{T},s}(q). \tag{3.2}$$

The right-hand side of (3.2) can be interpreted as a polynomial (with complex coefficients) evaluated at q. Let $\mathbf{H}$ be an F-stable maximal torus contained in an F-stable Borel subgroup of $\mathbf{Z}(s)^0$. Then

$$\begin{aligned} \operatorname{St}(s) f_{\mathbf{T},s}(1/q) &= \operatorname{St}(s)\, \lambda_{\mathbf{H}}(1/q)\, P_{\mathbf{Z}(s)^0}(1/q)\, \lambda_{\mathbf{T}}(1/q)^{-1} \\ &= \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{G}} \lambda_{\mathbf{H}}(q)\, P_{\mathbf{Z}(s)^0}(q)\, \lambda_{\mathbf{T}}(q)^{-1} \\ &= \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{G}} f_{\mathbf{T},s}(q), \end{aligned}$$

since $\dim \mathbf{H} = \dim \mathbf{T}$. We now need the following result, first communicated to the author by G. Lusztig (private communication).

(3.3) THEOREM (Lusztig). *If T is an F-stable maximal torus and θ: $\mathbf{T}^F \to C^*$ is a linear character, then $(R_{\mathbf{T}}^{\mathbf{G}}\theta)^* = \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{G}} R_{\mathbf{T}}^{\mathbf{G}}\theta$.*

Combining Theorem 3.3 with the observations above, we have proved the next theorem.

(3.4) THEOREM. *Let $s \in \mathbf{G}^F$ be semisimple and let $\chi \in \operatorname{char} \mathbf{G}^F$. Then $\chi^*(s) = \operatorname{St}(s)\, \chi^{\#}(s)$, where $\chi^{\#}(s)$ is obtained from expression* (3.2) *for* $\chi(s)$ *by replacing q by $1/q$.*

Now consider the case $s = 1$. Formula (3.2) takes the following form (see Lusztig [19, 3.14]).

$$\chi(1) = \sum_{(\mathbf{T},\theta)} \varepsilon_{\mathbf{T}}\varepsilon_{\mathbf{G}} \frac{(\chi, R_{\mathbf{T}}^{\mathbf{G}}\theta)}{(R_{\mathbf{T}}^{\mathbf{G}}\theta, T_{\mathbf{T}}^{\mathbf{G}}\theta)} \frac{|\mathbf{G}^F|_{p'}}{|\mathbf{T}^F|}. \tag{3.5}$$

The sum in (3.5) is over the $\mathbf{G}^F$-classes of pairs $(\mathbf{T}, \theta)$. The right-hand side of (3.5) is viewed as the value $d_\chi(q)$ of a polynomial $d_\chi(u)$ with rational coefficients ($\chi \in \operatorname{char} \mathbf{G}^F$).

(3.6) COROLLARY. *If $\chi \in \operatorname{char} \mathbf{G}^F$, then $d_{\chi^*}(q) = q^{|\Phi^+|}\, d_\chi(1/q)$, where Φ is the root system of* $\mathbf{G}$.

Let Φ be a fixed root system. If we consider all connected, reductive $\mathbf{G}$ defined over finite fields with root system Φ, then the resulting set of polynomials $d_\chi(u)$, $\chi \in \operatorname{irr} \mathbf{G}^F$, is finite [11, Proposition 6.3 and Theorem 6.8]. If we further assume $\mathbf{G}$ is semisimple, then there are integers M, N, and E (depending only on Φ) such that $|\mathbf{G}^F|$ always divides $q^M(q^N - 1)^E$ (see Steinberg [23, 11.16]). Also, if $\chi \in \operatorname{irr} \mathbf{G}^F$, then $\chi(1)$ divides $|\mathbf{G}^F|$ [10, 33.7]. Thus if we assume q is large (with respect to Φ) and $\mathbf{G}$ is semisimple, the polynomials $d_\chi(u)$, $\chi \in \operatorname{irr} \mathbf{G}^F$, can be factored as $d_\chi(u) = au^e\phi(u)$ with a rational, e a nonnegative integer and $\phi(u)$ a product of cyclotomic polynomials. This fact, together with Corollary 3.6, proves the following.

(3.7) COROLLARY. *Let Φ be a root system. Assume* $\mathbf{G}$ *is a semisimple group defined over F_q, with root system Φ and Frobenius endomorphism F. If q is large with respect to Φ, and if $\chi \in \operatorname{irr} \mathbf{G}^F$, then $\chi^*(1)/\chi(1)$ is of the form $\pm q^n$ for some integer n.*

4. DUALITY AND CHARACTER DEGREES

From now on, $\mathbf{G}$ will denote an algebraic group which is defined over a finite field $\mathbb{F}_q$, with corresponding Frobenius endomorphism F. It will be assumed that $\mathbf{G}$ is at least connected and reductive. Let $V(1)$ be the set of unipotent element of $\mathbf{G}^F$. We will consider the following two statements about the pair $(\mathbf{G}, F)$.

(4.1) For any $\chi \in \operatorname{irr} \mathbf{G}^F$, the quantity $C(\chi) = \sum_{u \in V(1)} \chi(u)/\chi(1)$ is of the form $\pm q^e$ for some nonnegative integer e.

(4.2) For any $\chi \in \operatorname{irr} \mathbf{G}^F$, the quantity $\chi^*(1)/\chi(1)$ is of the form $\pm q^n$ for some integer n.

For $\chi \in \operatorname{irr} \mathbf{G}^F$, let $\chi' \in \operatorname{irr} \mathbf{G}^F$ be such that $\chi^* = \pm\chi'$ (Theorem 1.5). Then $C(\chi)$ and $C(\chi')$ are algebraic integers [10, p. 236]. Also, $C(\chi) = |\mathbf{G}^F|_p \chi^*(1)/\chi(1)$ and $C(\chi') = |\mathbf{G}^F|_p \chi(1)/\chi^*(1)$ by Corollary 2.6, since $\chi^{**} = \chi$ (Theorem 1.5). Therefore $C(\chi)$ is a rational integer, and $\chi(1)_{p'} = |\chi^*(1)|_{p'}$. Since $|\mathbf{G}^F|_p = q^{|\Phi^+|}$, where Φ is the root system of $\mathbf{G}$, we have proved the following.

(4.3) Proposition. *The pair* $(\mathbf{G}, F)$ *satisfies* (4.1) *if and only if* $(\mathbf{G}, F)$ *satisfies* (4.2).

By Corollary 3.7, we know that (4.1) and (4.2) are valid for $(\mathbf{G}, F)$ if $\mathbf{G}$ is semisimple and q is sufficiently large with respect to Φ.

The proof of the next lemma is straightforward and will be omitted.

(4.4) Lemma. *Let $G = \langle B, N \rangle$ be a finite group with split (B, N)-pair of characteristic p, and assume H is a subgroup of $T \cap Z(G)$, $T = B \cap N$. Let $\bar{G} = G/H$, and give $\bar{G}$ the natural induced split (B, N)-pair structure. For $\chi \in \operatorname{char} \bar{G}$, let $L\chi \in \operatorname{char} G$ be the lift of χ to G via the projection $G \to \bar{G}$, so that $L\chi(g) = \chi(gH)$. Then $(L\chi)^* = L(\chi^*)$. Thus if $\chi \in \operatorname{irr} \bar{G}$, $\chi^*(1)/\chi(1) = (L\chi)^*(1)/L\chi(1)$.*

(4.5) Lemma. *Suppose that G and G' are finite groups with split (B, N)-pairs of characteristic p, that $G \trianglelefteq G'$ and that G and G' have a common Coxeter system (W, R). For $J \subseteq R$ let P_J (P'_J) be the corresponding standard parabolic subgroup of G (G', respectively). Assume $O_p(P_J) = O_p(P'_J)$ for $J \subseteq R$. Let $\chi \in \operatorname{irr} G'$, and let $\chi_0 \in \operatorname{irr} G$ be any component of the restriction $\chi|_G$ of χ to G. Then $\chi^*(1)/\chi(1) = \chi_0^*(1)/\chi_0(1)$.*

Proof. First note that

$$\chi^*(1) = \sum_{J \subseteq R} (-1)^{|J|} |G' : P'_J| (\chi|_{V'_J}, 1_{V'_J})_{V'_J},$$

where $V'_J = O_p(P'_J)$. Also, $V'_J = O_p(P_J) = V_J$ for $J \subseteq R$. Thus $U = V_\phi = V'_\phi = U'$, so that G contains all p-elements of G'. If $J \subseteq R$, then $GP'_J = P'_K$ for some $K \subseteq R$. Then $V'_K \subseteq O_p(G) = V_R = \{1\}$, and therefore $K = R$ and $GP'_J = G'$. Also, $P_J = N_G(V_J) = G \cap N_{G'}(V'_J)$ (Curtis [8, Proposition 1.5]), and therefore $|G : P_J| = |G' : P'_J|$, for $J \subseteq R$. It follows that $\chi^*(1) = (\chi|_G)^*(1)$.

By Clifford's theorem ([10, 49.7]) we have $\chi|_G = m(\chi_0 + \chi_1 + \cdots + \chi_{k-1})$, where $m \geqslant 1$ and $\chi_0, \ldots, \chi_{k-1}$ are distinct G'-conjugate irreducible characters of G. Thus $\chi_0(1) = \cdots = \chi_{k-1}(1)$ and $\chi(1) = mk\chi_0(1)$. Since $G \trianglelefteq G'$, the set $V(1)$ of unipotent elements of G is G'-invariant. Therefore $\sum_{u \in V(1)} \chi_0(u) = \cdots = \sum_{u \in V(1)} \chi_{k-1}(u)$. From this it follows that $\chi_0^*(1) = \cdots =$

$\chi_{k-1}^*(1)$ by Corollary 2.6. Therefore $\chi^*(1)=(\chi|_G)^*(1)=mk\chi_0^*(1)$, and the proof of the lemma is completed.

Let π: $\tilde{\mathbf{G}} \to \mathbf{G}$ be an isogeny of semisimple groups. Assume $\tilde{\mathbf{G}}$, $\mathbf{G}$ are defined over $\mathbb{F}_q$, with Frobenius endomorphisms $\tilde{F}$, F, and that $\pi\tilde{F}=F\pi$ (i.e., π is defined over $\mathbb{F}_q$). Let $\tilde{G}=\tilde{\mathbf{G}}^{\tilde{F}}$, $G=\mathbf{G}^F$ and $\tilde{H}=(\ker \pi)^{\tilde{F}}$. The next lemma follows from Lemmas 4.4 (applied to $\tilde{G}\to\tilde{G}/\tilde{H}$) and 4.5 (applied to $\tilde{G}/\tilde{H}\cong\pi(\tilde{G})\trianglelefteq G$).

(4.6) LEMMA. *Assume π: $\tilde{\mathbf{G}}\to\mathbf{G}$ is an isogeny of semisimple groups defined over $\mathbb{F}_q$. If* (4.2) *holds for* $(\tilde{\mathbf{G}}, \tilde{F})$, *then* (4.2) *holds for* $(\mathbf{G}, F)$.

(4.7) LEMMA. *Assume that* $\mathbf{G}$ *is connected, reductive and defined over* F_q, *with Frobenius endomorphism* F. *Let* $\mathbf{G}_0$ *be a closed, connected, reductive* F-*stable subgroup of* $\mathbf{G}$, *and assume* $D\mathbf{G}_0=D\mathbf{G}$. *Then* (4.2) *holds for* $(\mathbf{G}, F)$ *if and only if* (4.2) *holds for* $(\mathbf{G}_0, F)$.

Proof. Apply Lemma 4.5 to $\mathbf{G}_0^F\trianglelefteq\mathbf{G}^F$.

We now show that (4.2) will hold in general if (4.2) is true for certain special cases (see Lusztig [16, 1.18] for a similar reduction argument). To prove that (4.2) is true for $(\mathbf{G}, F)$, it is enough to prove (4.2) holds for $(D\mathbf{G}, F)$ (Lemma 4.7). Thus we may assume $\mathbf{G}=D\mathbf{G}$ is semisimple. Let $\mathbf{G}_1,\dots,\mathbf{G}_m$ be the simple components of $\mathbf{G}$ (see Humphreys [14, 27.5]). Let $\mathbf{H}_1,\dots,\mathbf{H}_k$ be the subgroups of $\mathbf{G}$ corresponding to the orbits of F on $\{\mathbf{G}_1,\dots,\mathbf{G}_m\}$. Then $F\mathbf{H}_i=\mathbf{H}_i$, and multiplication $\mathbf{H}_1\times\cdots\times\mathbf{H}_k\to\mathbf{G}$ is an isogeny defined over $\mathbb{F}_q$. By Lemma 4.6, we may reduce to the case $\mathbf{G}=\mathbf{H}_1$, and may assume $F\mathbf{G}_i=\mathbf{G}_{i+1}$ $(1\leqslant i<m)$, $F\mathbf{G}_m=\mathbf{G}_1$. Let $\tilde{\mathbf{G}}=\mathbf{G}_1\times\cdots\times\mathbf{G}_m$, and define $\tilde{F}$: $\tilde{\mathbf{G}}\to\tilde{\mathbf{G}}$ by $\tilde{F}(g_1,\dots,g_m)=(Fg_m, Fg_1,\dots,Fg_{m-1})$. Then $\tilde{F}$ is the Frobenius endomorphism for an $\mathbb{F}_q$-rational structure on $\tilde{\mathbf{G}}$, and multiplication $\tilde{\mathbf{G}}\to\mathbf{G}$ is an isogeny defined over $\mathbb{F}_q$. Applying Lemma 4.6 again, we may assume $\mathbf{G}=\mathbf{G}_1\times\cdots\times\mathbf{G}_m$ and that $F\mathbf{G}_i=\mathbf{G}_{i+1}$ $(1\leqslant i<m)$, $F\mathbf{G}_m=\mathbf{G}_1$. Note that $\mathbf{G}_1$ is defined over $\mathbb{F}_{q^m}$, with Frobenius endomorphism F^m. Also, $\mathbf{G}_1^{F^m}\cong\mathbf{G}^F$. Therefore (4.2) holds for $(\mathbf{G}, F)$ if (4.2) holds for $(\mathbf{G}_1, F^m)$ (with q^m replacing q). We thus reduce further to the case $\mathbf{G}=\mathbf{G}_1$, so that the Dynkin diagram Γ for $\mathbf{G}$ is connected. Applying Lemma 4.6 once more, we may also assume that $\mathbf{G}$ is simply connected. The reduction is summarized in the following theorem.

(4.7) THEOREM. *Let* $\Gamma_1,\dots,\Gamma_k$ *be connected Dynkin diagrams, and let* q *be a prime power. Assume that* (4.2) *holds for* $(\tilde{\mathbf{G}}, \tilde{F})$ *whenever* $\tilde{\mathbf{G}}$ *is a simply connected group defined over* $\mathbb{F}_{q^m}$, $m\geqslant 1$, *with Frobenius endomorphism* $\tilde{F}$ *and Dynkin diagram* Γ_i *for some* i. *Let* $\mathbf{G}$ *be a connected, reductive group defined over* F_q *with Frobenius endomorphism* F, *such that the components of the Dynkin diagram* Γ *of* $\mathbf{G}$ *are among* $\Gamma_1,\dots,\Gamma_k$. *Then* (4.2) *holds for* $(\mathbf{G}, F)$.

(4.8) THEOREM. *Let* $\mathbf{G}$ *be simply connected and defined over* $\mathbb{F}_q$, *with Frobenius endomorphism F. Assume* $(\mathbf{G}, F)$ *is of type* A_n, ${}^2A_n$, B_n, C_n, D_n *or* ${}^2D_n$. *Then* (4.2) *holds for* $(\mathbf{G}, F)$.

Proof. By Lemma 4.7, it is sufficient to show (4.2) holds when $\mathbf{G}$ is reductive, $D\mathbf{G}$ is simply connected, $Z(\mathbf{G})$ is connected, and $(\mathbf{G}, F)$ is of type A_n, ${}^2A_n$, B_n, C_n, D_n or ${}^2D_n$. Let $W = \mathbf{W}^F$ be the (relative) Weyl group of $G = \mathbf{G}^F$, with set of distinguished generators R. Suppose $\chi \in \operatorname{irr} G$, and let $\chi' \in \operatorname{irr} G$ be such that $\chi' = \pm\chi^*$ (Theorem 1.5). By the remarks after Theorem 1.5, χ and χ' are both components of some induced cuspidal representation $I_J\varphi$, $J \subseteq R$. Let $W(\varphi)$ be the set of distinguished (W_J, W_J)-double coset representatives w such that ${}^wJ = J$ and ${}^w\varphi = \varphi$. Thus $W(\varphi)$ is isomorphic to the stabilizer of φ in $N_W(W_J)/W_J$. Also, $W(\varphi)$ is a Coxeter group, with set of distinguished generators S, and the irreducible components of $I_J\varphi$ correspond by specialization to the irreducible characters of $W(\varphi)$ (see Asai [4, Sect. 7]). Let χ correspond to μ and χ' to μ', $\mu, \mu' \in \operatorname{irr} W(\varphi)$. Assume $|S| \geqslant 1$ (if $W(\varphi) = \{1\}$, then $\chi = \chi' = \pm\chi^*$; so $\chi^*(1)/\chi(1) = \pm 1$ and the assertion of (4.2) holds for $\mathbf{G}$, F, χ). We shall prove $\mu' = \operatorname{sgn} \otimes \mu$, where sgn is the sign character of $(W(\varphi), S)$.

Assume $|S| = 1$, so that $|W(\varphi)| = 2$ and $I_J\varphi = \chi + \chi_1$, $\chi_1 \in \operatorname{irr} G$, $\chi_1 \neq \chi$. We must show $\chi' = \chi_1$. By Curtis's result (Theorem 1.4), Lemma 1.2 and [4, 1.3.18], we may assume J is a maximal subset of R. Then by [8, Theorem 3.5], we have $\chi^* = (-1)^{|R|-1} m I_J\varphi + (-1)^{|R|}\chi$, for some $m > 0$. Thus $m(\chi + \chi_1) = \pm\chi \pm \chi^*$, and so $\chi_1 = \chi' = \pm\chi^*$ since $\chi_1 \neq \chi$. It follows that $\mu' \neq \mu$, and therefore $\mu' = \operatorname{sgn} \otimes \mu$ in this case.

Assume that $|S| > 1$. By a suitable induction hypothesis, Theorem 1.4 and Asai [4, Theorem 7.2.5], we may assume that $(\mu', \operatorname{Ind}_{W(\varphi)_K}^{W(\varphi)} \nu) = (\operatorname{sgn} \otimes \mu, \operatorname{Ind}_{W(\varphi)_K}^{W(\varphi)} \nu)$ when $K \subseteq S$, $K \neq S$, and $\nu \in \operatorname{irr} W(\varphi)_K$, where $W(\varphi)_K$ is generated by K. It follows that $\mu' = \operatorname{sgn} \otimes \mu$ (see Benson and Curtis [5]), since $|S| > 1$.

The irreducible characters of $G = \mathbf{G}^F$ are partitioned into geometric conjugacy classes $\operatorname{irr}(\mathbf{G}^F, [s])$, one for each conjugacy class $[s]$ of semisimple elements of $\mathbf{G}^{*F}$, where $\mathbf{G}^*$ is the dual group of $\mathbf{G}$ (Deligne and Lusztig [11, Sects. 5 and 6]). Let $\chi \in \operatorname{irr}(\mathbf{G}^F, [s])$. Then $\chi' \in \operatorname{irr}(\mathbf{G}^F, [s])$ by the remarks above and Asai [4, Theorem 6.1]. Let R: $\operatorname{irr}(\mathbf{G}^F, [s]) \to \operatorname{irr}(Z_{\mathbf{G}^*}(s)^{*F}, [1])$ be the bijection defined in Lusztig [18] and Asai [4], and set $\rho = R(\chi)$, $\rho' = R(\chi')$. Then $\chi'(1)/\chi(1) = \rho'(1)/\rho(1)$ (see [18, Theorem 8.2; 4, Theorem 6.1]). Also, ρ and ρ' are both components of the character induced from some irreducible cuspidal unipotent character ρ_1 of some Levi factor of $Z_{\mathbf{G}^*}(s)^{*F}$. By Lusztig [19, 3.26.1], $\rho'(1)/\rho(1) = d(\chi_{\mu'})/d(\chi_\mu)$, where $d(\chi_\mu)$ and $d(\chi_{\mu'})$ are specializations of the generic degrees for the characters of the generalized Hecke algebra of $I_J\varphi$ which correspond to μ and μ'. Since $\mu' = \operatorname{sgn} \otimes \mu$, $d(\chi_{\mu'})$ can be obtained from $d(\chi_\mu)$ by replacing q by $1/q$ and multiplying by a power of q (see Green [12]). Hence $d(\chi_{\mu'})/d(\chi_\mu)$ is of the form q^n for some integer n, by the formulas of Hoefsmit [13]. Therefore $\chi^*(1)/\chi(1) = \pm\chi'(1)/\chi(1) = \pm\rho'(1)/\rho(1) = \pm q^n$; so that (4.2) holds for $\mathbf{G}, F, \chi$. This completes the proof of the theorem.

If the characteristic p of $\mathbb{F}_q$ is not 2, then a simpler proof of Theorem 4.8 is possible. Let $\mathbf{G}$ be as in the proof above. The formulas of Lusztig [18, 19], Asai [4], and Hoefsmit [13] show that $\chi(1)_p$ is always a power of q when $\chi \in \operatorname{irr} \mathbf{G}^F$. Thus $\chi^*(1)/\chi(1)$ is trivially of the form $\pm q^n$ in this case, since $\pm\chi^* \in \operatorname{irr} \mathbf{G}^F$.

If $\mathbf{G}$ is simply connected and $(\mathbf{G}, F)$ is of type ${}^3D_4$, E_6, ${}^2E_6$, E_7, E_8, F_4 or G_2, then (4.2) holds for $(\mathbf{G}, F)$ provided q is sufficiently large (Corollary 3.7). Combining this with Theorems 4.7 and 4.8 proves the following.

(4.9) THEOREM. *Assertion* (4.2) *holds for any* $(\mathbf{G}, F)$, *provided* q *is large.*

Note that the sense in which q must be large in Theorem 4.9 does not depend on the particular root system of $\mathbf{G}$ (as in Corollary 3.7). It is expected that (4.2) is true in general.

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